Reasoning about dependence and independence in aggregation problems

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Social Choice Theory

"Social choice theory is the study of collective decision processes and procedures. It is not a single theory, but a cluster of models and results concerning the aggregation of individual inputs (e.g., votes, preferences, judgments, welfare) into collective outputs (e.g., collective decisions, preferences, judgments, welfare)."

C. List. *Social Choice Theory*. Stanford Encyclopedia of Philosophy, 2013.

Two Desiderata

1. The group decision should *depend* in the right way on the voters' opinions.

2. Each voter's choice is *independent* of the other voters' choices.

Preliminaries

Alternatives, Voters, Opinions

X is a set with $|X| \geq 3$, elements of which are called **alternatives**

V is a nonempty set, elements of which are called **voters**

P (possibly with subscripts) is a relation on X that is intended to represent a voter's or society's opinion about the alternaties

Preferences, I

A binary relation P on X is a **strict weak order** if and only if P satisfying the following conditions:

- **irreflexivity**: *not* xPx;
- **quasi-transitivity**: if xPy and yPz, then xPz;
- **negative transitivity**: if xPy, then for all $z \in X$, either xPz or zPy

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Note: In the Social Choice literature, sometimes P (strict preference) and I (indifference) are taken as primitive, other times R (weak preference) is taken as primitive.

Preferences, II

We adopt the following notation for classes of binary relations on X:

- $P(X)$ is the set of all irreflexive binary relations on X.
- $O(X)$ is the set of all strict weak orders on X.
- $L(X)$ is the set of all linear orders on X.

Profiles

A **profile** $\mathbf{P}$ is an element of $O(X)^V$, i.e., a function assigning to each $i \in V$ a relation $P_i \in O(X)$. For $x, y \in X$, let:

$$\mathbf{P}(x, y) = \{i \in V \mid xP_iy\};$$

$$\mathbf{P}_{|\{x,y\}} = \text{the function assigning to each } i \in V \text{ the relation } P_i \cap \{x, y\}^2.$$

Collective Choice Rule

A **collective choice rule** (CCR) is a function f from a subset of $O(X)^V$ to $P(X)$.
We write $xf(\mathbf{P})y$ to mean $\langle x, y\rangle \in f(\mathbf{P})$.

From Social Choice to Dependence Logic

EP and Fan Yang. *Dependence and Independence in Social Choice: Arrow's Theorem*. in *Dependence Logic*, pgs. 235 - 260, 2016.

From Social Choice to Dependence Logic

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For $\mathbf{P} = (P_1, \ldots, P_n) \in dom(f)$, $s_{\mathbf{P},f} : Vars \rightarrow P(x)$ is a substitution defined as follows:

$$s_{\mathbf{P},f}(x_1) = P_1, \ \ldots, \ s_{\mathbf{P},f}(x_n) = P_n \text{ and } s_{\mathbf{P},f}(y) = f(\mathbf{p}).$$

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A CCR f is associated with a team:

$$X_f = \{s_{\mathbf{P},f} \mid \mathbf{P} \in dom(f)\}$$

Example

	x_1	x_2	y
s_1	$a\ b\ c$	$c\ b\ a$	$b\ a\ c$
s_2	$a\ c\ b$	$b\ c\ a$	$c\ b\ a$
s_3	$c\ a\ b$	$b\ a\ c$	$a\ c\ b$
s_4	$b\ c\ a$	$a\ c\ b$	$c\ a\ b$
s_5	$a\ b\ c$	$b\ c\ a$	$b\ a\ c$

$P_{ab}(x)$ is true at s provided that $\langle a, b\rangle \in s(x)$ (i.e., $a\ s(x)\ b$)

Example

	x_1	x_2	y
s_1	$a\ b\ c$	$c\ b\ a$	$b\ a\ c$
s_2	$a\ c\ b$	$b\ c\ a$	$c\ b\ a$
s_3	$c\ a\ b$	$b\ a\ c$	$a\ c\ b$
s_4	$b\ c\ a$	$a\ c\ b$	$c\ a\ b$
s_5	$a\ b\ c$	$b\ c\ a$	$b\ a\ c$

s_1 makes $P_{cb}(x_2)$ true

Rationality Postulates

$$
\begin{aligned}
\Gamma_{Ord} := & \{\forall x \neg (P_{ab}(x) \wedge P_{ba}(x)) \mid a, b \in X \text{ and } a \neq b\} \\
& \cup \{\forall x (P_{ab}(x) \wedge P_{bc}(x) \supset P_{ac}(x)) \mid a, b, c \in X\} \\
& \cup \{\forall x (P_{ab}(x) \vee P_{ba}(x))) \mid a, b \in X\}
\end{aligned}
$$

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Functional Dependence

$$\theta_F := \; =\!(x_1, \ldots, x_n, y)$$

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$$\theta_F := \bigwedge_{a,b \in X} =\!(\langle P_{cd}(x_i) \mid c, d \in X \text{ and } 1 \leq i \leq n \rangle, P_{ab}(y))$$

$$\exists^1 x\varphi \; := \; \exists x(\,{=}(x) \wedge \varphi).$$

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$$=\!(\alpha_1, \ldots, \alpha_k, \beta)$$

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$$
\begin{aligned}
&=(\alpha_1, \ldots, \alpha_k, \beta) \; := \\
&\exists u_1 \ldots \exists u_k \exists u \, \exists^1 z_0 \, \exists^1 z_1 \Big(=(x_1, \ldots, x_k, u) \wedge (z_0 \neq z_1) \\
&\wedge \bigwedge_{i=1}^{k} ((\alpha_i \to u_i = z_1) \wedge (\neg\alpha_i \to u_i = z_0)) \\
&\wedge (\beta \to u = z_1) \wedge (\neg\beta \to u = z_0) \Big)
\end{aligned}
$$

where $u_1, \ldots, u_k, u, z_0, z_1$ are fresh variables.

Example

	x_1	x_2	y
s_1	$a \; b \; c$	$c \; b \; a$	$b \; a \; c$
s_2	$a \; c \; b$	$b \; c \; a$	$c \; b \; a$
s_3	$c \; a \; b$	$b \; a \; c$	$a \; c \; b$
s_4	$b \; c \; a$	$a \; c \; b$	$c \; a \; b$
s_5	$a \; b \; c$	$b \; c \; a$	$b \; a \; c$

Weak Pareto

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$$\theta_{WP} \quad := \quad \bigwedge_{i \in V} P_{ab}(x_i) \rightarrow P_{ab}(y)$$

Decisiveness

$$\delta_{ab}(x_{i_1}, \ldots, x_{i_k}) := (P_{ab}(x_{i_1}) \wedge \cdots \wedge P_{ab}(x_{i_k})) \rightarrow P_{ab}(u).$$

$$\delta(x_{i_1}, \ldots, x_{i_k}) := \bigwedge_{a,b \in X} \delta_{ab}(x_{i_1}, \ldots, x_{i_k}).$$

$$\theta_{WP} := \delta(x_1, \ldots, x_n).$$

Decisive Coalitions, continued

Let f be a CCR. For any $x, y \in X$ and $A \subseteq V$:

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1. A is **almost decisive for** x **over** y **according to** f iff for any $\mathbf{P} \in \text{dom}(f)$, if $A = \mathbf{P}(x, y)$ and $A^c = \mathbf{P}(y, x)$, then $xf(\mathbf{P})y$;

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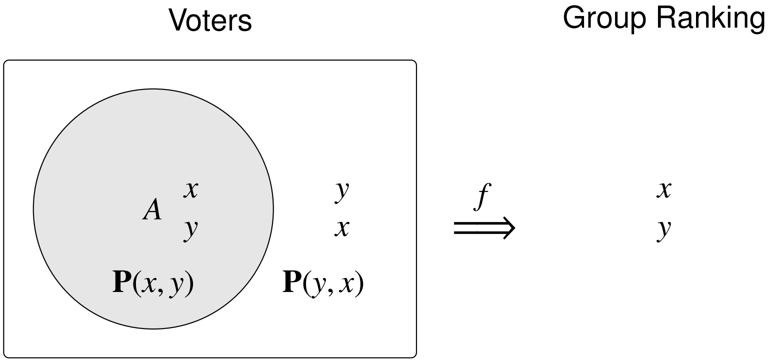
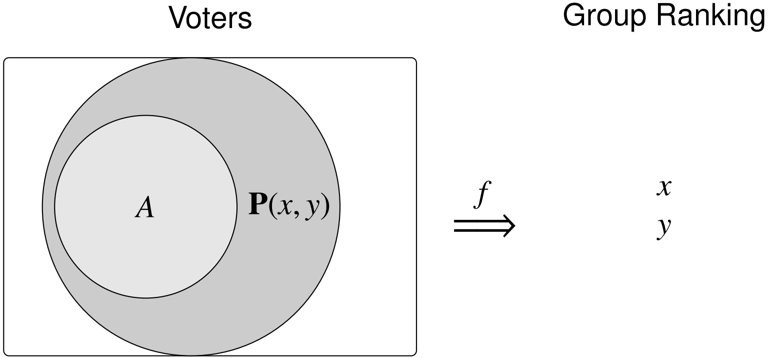
2. A is **decisive for** x **over** y **according to** f iff for any $\mathbf{P} \in \mathrm{dom}(f)$, if $A \subseteq \mathbf{P}(x, y)$, then $xf(\mathbf{P})y$.

Almost Decisive



Decisive



Example

	x_1	x_2	y
s_1	a b c	c b a	b a c
s_2	a c b	b c a	c b a
s_3	c a b	b a c	a c b
s_4	b c a	a c b	c a b
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s_5	$a\ b\ c$	$b\ c\ a$	$b\ a\ c$

Independence of Irrelevant Alternatives

IIA: The group's ranking of a and b should only depend on the voter's rankings of a and b

for all profiles $\mathbf{P}, \mathbf{P}'$ if for all $i \in V$, $\mathbf{P}_i|_{\{a,b\}} = \mathbf{P}'_i|_{\{a,b\}}$, then $f(\mathbf{P})|_{\{a,b\}} = f(\mathbf{P}')|_{\{a,b\}}$

$\theta_{IIA} \;:=\; =(P_{ab}(x_1), \ldots, P_{ab}(x_n), P_{ab}(y))$

	x_1	x_2	y
s_1	$a\ b\ c$	$c\ b\ a$	$b\ c\ a$
s_2	$a\ c\ b$	$b\ c\ a$	$c\ b\ a$
s_3	$c\ a\ b$	$b\ a\ c$	$c\ b\ a$
s_4	$b\ c\ a$	$a\ c\ b$	$b\ c\ a$

The voters are free to adopt any ranking *and* the voters' opinions are independent of each other.

Domain Conditions

Universal Domain: The domain of the social welfare (choice) function is $L(X)^n$ (or $O(X)^n$)

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Epistemic Rationale: “If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings.” (Arrow, 1963, pg. 24)

1. Voters are free to choose any ranking.
2. The voters’ choices are independent.

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2. The voters’ choices are independent.

Both assumptions are used.

	x_1	x_2	y
s_1	a b c	c b a	b c a
s_2	a c b	b c a	c b a
s_3	c a b	b a c	c b a
s_4	b c a	a c b	b c a

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s_1	a b c	c b a	b c a
s_2	a c b	b c a	c b a
s_3	c a b	b a c	c b a
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	x_1	x_2	y
s_1	$a\ b\ c$	$c\ b\ a$	$b\ c\ a$
s_2	$a\ c\ b$	$b\ c\ a$	$c\ b\ a$
s_3	$c\ a\ b$	$b\ a\ c$	$c\ b\ a$
s_4	$b\ c\ a$	$a\ c\ b$	$b\ c\ a$
s_5	...	$c\ a\ b$	...

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s_1	a b c	c b a	b c a
s_2	a c b	b c a	c b a
s_3	c a b	b a c	c b a
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s_5	c a b	c a b	c a b

	x_1	x_2	y
s_1	a b c	c b a	b c a
s_2	a c b	b c a	c b a
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s_5	c a b	c a b	c a b

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s_3	$c\ a\ b$	$b\ a\ c$	$c\ b\ a$
s_4	$b\ c\ a$	$a\ c\ b$	$b\ c\ a$
s_5	$c\ a\ b$	$c\ a\ b$	$c\ a\ b$
s_6	$a\ b\ c$	$c\ a\ b$	???

	x_1	x_2	y
s_1	a b c	c b a	b c a
s_2	a c b	b c a	c b a
s_3	c a b	b a c	c b a
s_4	b c a	a c b	b c a
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s_4	$b \; c \; a$	$a \; c \; b$	$b \; c \; a$
s_5	$c \; a \; b$	$c \; a \; b$	$c \; a \; b$
s_6	$a \; b \; c$	$c \; a \; b$	$b \; c, c \; a$

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s_5	$c\ a\ b$	$c\ a\ b$	$c\ a\ b$
s_6	$a\ b\ c$	$c\ a\ b$	$b\ c\ a$

(*All Rankings*) $\theta_{AR} := \bigwedge_{i=1}^{n} \bigwedge_{a_1 \cdots a_k \in L(X)} \exists x (P_{a_1 \cdots a_k}(x) \wedge x \subseteq x_i)$

(*Independence*) $\theta_I := \bigwedge_{i=1}^{n} x_i \perp \langle x_j \mid j \neq i \rangle$

Dictatorship

$$\theta_D := \bigvee_{i=1}^{n} \delta(v_i).$$

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$$\varphi \vee\!\!\!\vee \psi \equiv \exists^1 x\, \exists^1 y \Big(((x = y) \vee \varphi) \wedge ((x \neq u) \vee \psi) \Big)$$

where x, u are fresh variables.

Arrow's Theorem

Theorem (Arrow, 1952) Suppose that the set of voters V is finite and there are at least three alternatives ($|X| \geq 3$).

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Arrow's Theorem

Theorem (Arrow, 1952) Suppose that the set of voters V is finite and there are at least three alternatives ($|X| \geq 3$). Then, a collective choice rule f satisfies universal domain (UD), weak Pareto (WP), independence of irrelevant alternatives (IIA) and full rationality (FR) if, and only if, f has a **dictator**.

Deriving Arrow's Theorem

$FO(\perp, \subseteq)$

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$\Gamma_{Ord}, \theta_{AR}, \theta_I, \theta_{WP}, \theta_{IIA}, \sim \theta_D \vdash \perp$

M. Hannula. *Axiomatizing first-order consequences in independence logic*. Annals of Pure and Applied Logic, 166(1):61 – 91, 2015.

F. Yang. *Negation and Partial Axiomatizations of Dependence and Independence Logic Revisited*. Annals of Pure and Applied Logic, 2018.

Arrow’s Theorem

Weaken Pareto

Wilson's Theorem (1972)

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Weaken Rationality Postulates

Interval Orders, Semi-Orders, Choice functions

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Infinite Voters, Judgement Aggregation,
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Generalizations
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Domain Restrictions

- Single-Peaked preferences
- Sen's Value Restriction
- Assumptions about the distribution of preferences

W. Gaertner. *Domain Conditions in Social Choice Theory*. Cambridge University Press, 2001.

A. Sen. *A Possibility Theorem on Majority Decisions*. Econometrica 34, 1966, pgs. 491 - 499.

M. Regenwetter, B. Grofman, A.A.J. Marley and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

Saturated Domains

- A set $\{a, b, c\} \subseteq X$ is a **free triple** for a domain $\mathcal{D}$ provided that *every* profile over $\{a, b, c\}$ is *contained in* some profile in $\mathcal{D}$.
- $B = \{a, b\}$ and $C = \{c, d\}$ are **strongly connected** in $\mathcal{D}$ provided that $|B \cup C| = 3$ and $B \cup C$ is a free triple (and they are non-trivial).
- A set of profiles $\mathcal{D}$ is **saturated** provided there are at least two non-trivial pairs of candidates in $\mathcal{D}$ and every two non-trivial pairs of candidates are *connected*.

Theorem Arrow's Theorem holds on saturated domains.

E. Kalai, E. Muller, and M. Satterthwaite. *Social welfare functions when preferences are convex and continuous: Impossibility results*. Public Choice, 34, 87 - 97, 197.

$$=\!(x_1, \ldots, x_n, y)$$

$$\Downarrow \quad \text{(IIA+Trans)}$$

$$=\!(x_1, y) \mathbin{\mathbb{V}} =\!(x_2, y) \mathbin{\mathbb{V}} \cdots \mathbin{\mathbb{V}} =\!(x_n, y)$$

$$=\!(x_1, \ldots, x_n, y)$$

$$\Downarrow \quad \text{(IIA+Trans)}$$

$$=\!(x_1, y) \veebar \; =\!(x_2, y) \veebar \; \cdots \veebar \; =\!(x_n, y)$$

$$\Downarrow \quad \text{(LinOrders)}$$

$$x_1 = y \veebar x_2 = y \veebar \cdots \veebar x_n = y$$

(cf. *characterizations* of Arrovian dictators)

A. Rubinstein and P. Fishburn. *Algebraic Aggregation Theory*. Journal of Economic Theory, 38, pp. 63 - 77, 1986.

$$\langle J_1, J_2, \ldots, J_n \rangle \mapsto J$$

$$\begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix} \mapsto J$$

$$\begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix} \mapsto (y_1, y_2, \ldots, y_m)$$

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \mapsto (y_1, y_2, \ldots, y_m)$$

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Each x_{ij} is an element of a **field** B.

For $i = 1, \ldots, n$, $x_i = (x_{i1}, \ldots, x_{im})$ is an element of a **vector space** $X \subseteq B^m$ over B.

Aggregator: $f : X^n \to X$

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For $i = 1, \ldots, n$, $x_i = (x_{i1}, \ldots, x_{im})$ is an element of a **vector space** $X \subseteq B^m$ over B.

Aggregator: $f : X^n \to X$

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \mapsto f((x_{11}, \ldots, x_{1m}), (x_{21}, \ldots, x_{2m}), \ldots, (x_{n1}, \ldots, x_{nm}))$$

Each x_{ij} is an element of a **field** B.

For $i = 1, \ldots, n$, $x_i = (x_{i1}, \ldots, x_{im})$ is an element of a **vector space** $X \subseteq B^m$ over B.

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$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \mapsto (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$$

Each x_{ij} is an element of a **field** B.

For $i = 1, \ldots, n$, $x_i = (x_{i1}, \ldots, x_{im})$ is an element of a **vector space** $X \subseteq B^m$ over B.

Aggregator: $f : X^n \to X$

C1: $(x_{1j}, \ldots, x_{nj}) = (x'_{1j}, \ldots, x'_{nj})$ implies $f_j(x_1, \ldots, x_n) = f_j(x'_1, \ldots, x'_n)$

$$\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2j} & \cdots & x_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix} \mapsto (f_1(x_1, x_2, \ldots, x_n), \ldots, f_j(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$$

$$\begin{pmatrix} x'_{11} & \cdots & x'_{1j} & \cdots & x'_{1m} \\ x'_{21} & \cdots & x'_{2j} & \cdots & x'_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x'_{n1} & \cdots & x'_{nj} & \cdots & x'_{nm} \end{pmatrix} \mapsto (f_1(x'_1, x'_2, \ldots, x'_n), \ldots, f_j(x'_1, x'_2, \ldots, x'_n), \ldots, f_m(x'_1, x'_2, \ldots, x'_n))$$

C2: $(x_{1j}, \ldots, x_{nj}) = (b, \ldots, b)$ implies $f_j(x_1, \ldots, x_n) = b$

$$\begin{pmatrix} x_{11} & \cdots & b & \cdots & x_{1m} \\ x_{21} & \cdots & b & \cdots & x_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & b & \cdots & x_{nm} \end{pmatrix} \mapsto (f_1(x_1, x_2, \ldots, x_n), \ldots, b, \ldots, f_m(x_1, x_2, \ldots, x_n))$$

$$F_C = \{f \in F \mid f \text{ satisfies } C1 \text{ and } C2\}$$

$$F_C = \{f \in F \mid f \text{ satisfies } C1 \text{ and } C2\}$$

$$F_A = \{f \in F \mid f \text{ satisfies } C1 \text{ and } f_j(y+z) = f_j(y) + f_j(z) \text{ for all } j \leq m \text{ and column vectors for which } y, z, y+z \in X_j^n \}$$

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$$F_S = \{f \in F \mid f \text{ there exists } \lambda_1, \ldots, \lambda_n \in B \text{ such that } \sum \lambda_i = 1 \text{ and, for all } (x_1, \ldots, x_n) \in X^n, f(x_1, \ldots, x_n) = \sum \lambda_i x_i\}$$

$$F_C = \{f \in F \mid f \text{ satisfies } C1 \text{ and } C2\}$$

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$$F_S = \{f \in F \mid f \text{ there exists } \lambda_1, \ldots, \lambda_n \in B \text{ such that } \sum \lambda_i = 1 \text{ and, for all } (x_1, \ldots, x_n) \in X^n, f(x_1, \ldots, x_n) = \sum \lambda_i x_i\}$$

$$F_P = \{f \in F \mid f \text{ there is an } i = 1, \ldots, n \text{ such that for all } (x_1, \ldots, x_n) \in X^n \; f(x_1, \ldots, x_n) = x_i\}$$

Theorem 1. Suppose $m \geq 3$ and $X = \{(x^1, \ldots, x^m) \in B^m \mid \sum_j b_j x^j = b\}$ with $b_j \neq 0$ for all $j \leq m$. Then, $F_C \subseteq F_A$.

Theorem 1. Suppose $m \geq 3$ and $X = \{(x^1, \ldots, x^m) \in B^m \mid \sum_j b_j x^j = b\}$ with $b_j \neq 0$ for all $j \leq m$. Then, $F_C \subseteq F_A$.

Corollary 1. Suppose $m \geq 3$,
$X = \{(x^1, \ldots, x^m) \in \mathbb{R}^m \mid \sum_j b_j x^j = b$ and $x^j \geq 0$ for all j $\}$ with b and all b_j positive.
Then $F_C \subseteq F_A$.

Corollary 2. Give the hypothesis of Theorem 1, let $f \in F_C$. Then $f \in F_S$ if B is a finite field, or if $B = \mathbb{R}$ and every f_j is continuous or monotone.

Suppose that n experts are asked to submit their probability $p_i = (p_{i1}, \ldots, p_{im})$ over $m \geq 3$ mutually exclusive and exhaustive events.

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The aggregation for event j depends only on the experts' probabilities for event j

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The aggregation for event j depends only on the experts' probabilities for event j

If the aggregator satisfies $C1$ and $C2$, then Corollary 1 with $X = \{(p^1, \ldots, p^m) \mid p^j \geq 0 \text{ and } \sum p^j = 1\}$ implies that the aggregator is additive.

If it is also continuous, then Corollary 2 implies that the aggregator is a **weighted average** of the experts' probability vectors.

Aggregation Functions

Linear pooling: for all $A \in \mathcal{E}$, $f(\mathbf{P})(A) = w_1 P_1(A) + \cdots w_n P_n(A)$, with $\sum_i w_i = 1$

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Geometric pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

Aggregation Functions

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Geometric pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

Multiplicative pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)] \cdots [P_n(w)]$ with $c = \frac{1}{\sum_{w' \in W} [P_i(w')] \cdots [P_i(w')]}$

Note that multiplicative pooling = geometric pooling with weights all equal to 1.

Aggregating Probabilities

C. Genest and J. V. Zidek. *Combining probability distributions: A critique and an annotated bibliography*. Statistical Science,1(1), pp. 114 - 135, 1986.

F. Dietrich and C. List. *Probabilistic opinion pooling*. in Oxford Handbook of Probability and Philosophy, 2016.

Linear Pooling

J. Aczel and C. Wagner. *A characterization of weighted arithmetic means*. SIAM Journal on Algebraic and Discrete Methods 1(3), pp. 259 - 260, 1980.

K. J. McConway. *Marginalization and Linear Opinion Pools*. Journal of the American Statistical Association, 76(374), pp. 410 - 414, 1981.

Eventwise Independence For each event $A \in \mathcal{E}$, there exists a function $D_A : [0, 1]^n \to [0, 1]$ such that for each $\mathbf{P} = (P_1, \ldots, P_n)$,

$$f(\mathbf{P})(A) = D_A(P_1(A), \ldots, P_n(A))$$

Unanimity preservation For every profile $\mathbf{P} = (P_1, \ldots, P_n)$ in the domain of the aggregation function f, if all P_i are identical, then $f(\mathbf{P})$ is identical to them.

Theorem (Aczel and Wagner 1980; McConway 1981) Suppose $|W| > 2$. The linear pooling functions are the only eventwise-independent and unanimity-preserving aggregation functions (with domain $\mathcal{P}^n$).

Independence and Linear Pooling

K. Lehrer and C. Wagner. *Probability amalgamation and the independence issue: a reply to Laddaga*. Synthese 55, pp. 339 - 346, 1983.

C. Wagner. *On the Formal Properties of Averaging as a Method of Aggregation*. Synthese, 62, pp. 97 - 108, 1985.

C. Wagner. *Aggregating subjective probabilities: some limitative theorems*. Notre Dame Journal of Formal Logic, 25(3), pp. 233 - 240, 1984.

RI (Respect for Individual Attributions of Independence) For any propositions E and F and profile $\mathbf{P} = (P_1, \ldots, P_n)$, if $P_i(E \cap F) = P_i(E)P_i(F)$ for all $i = 1, \ldots, n$, the $f(\mathbf{P})(E \cap F) = f(\mathbf{P})(E)f(\mathbf{P})(F)$

RI (Respect for Individual Attributions of Independence) For any propositions E and F and profile $\mathbf{P} = (P_1, \ldots, P_n)$, if $P_i(E \cap F) = P_i(E)P_i(F)$ for all $i = 1, \ldots, n$, the $f(\mathbf{P})(E \cap F) = f(\mathbf{P})(E)f(\mathbf{P})(F)$

Theorem (Wagner). Suppose that $f : \mathcal{P}^n \to \mathcal{P}$. Then, f satisfies eventwise-independence, unanimity-preservation and respect for individual attributions of independence if, and only if, f is a dictatorship.

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} \\
p_4 &= w_1 p_{14} + w_2 p_{24} + \cdots + w_d p_{d4} + \cdots + w_n p_{n4} \\
&\vdots \\
p_k &= w_1 p_{1k} + w_2 p_{2k} + \cdots + w_d p_{dk} + \cdots + w_n p_{nk}
\end{aligned}
$$

$$
\begin{array}{c}
\\
1 \\
\vdots \\
d \\
\vdots \\
n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{array}{lclcl}
p_1 & = & w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} & & \\
p_2 & = & w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} & & \\
p_3 & = & w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} & & \\
p_4 & = & w_1 p_{14} + w_2 p_{24} + \cdots + w_d p_{d4} + \cdots + w_n p_{n4} & = & 0 \\
 & \vdots & & \vdots & \vdots \\
p_k & = & w_1 p_{1k} + w_2 p_{2k} + \cdots + w_d p_{dk} + \cdots + w_n p_{nk} & = & 0
\end{array}
$$

$$
\begin{array}{c}
 \\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2} \textstyle\sum_{j \neq d} w_j \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &= \frac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &= \frac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &= \frac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

For all i, $P_i(\{s_1, s_2\} \cap \{s_2, s_3\}) = P_i(\{s_1, s_2\})P_i(\{s_2, s_3\})$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $f(\mathbf{P})(\{s_1, s_2\} \cap \{s_2, s_3\}) = f(\mathbf{P})(\{s_1, s_2\}) f(\mathbf{P})(\{s_2, s_3\})$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c} 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\overset{\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array}}{\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}} \mapsto (p_1, \ldots, p_k)
$$

RI implies that $p_2 = (p_1 + p_2)(p_2 + p_3)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
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\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $\frac{1}{2} = (\frac{1}{2}(1 - w_d) + \frac{1}{2})(\frac{1}{2} + \frac{1}{2}w_d)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $2 = ((1 - w_d) + 1)(1 + w_d)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ d \\ \vdots \\ n \end{array} &
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix} \mapsto (p_1, \ldots, p_k)
\end{array}
$$

RI implies that $w_d = 0$ or $w_d = 1$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c c}
 & \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
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\begin{pmatrix}
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0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
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\end{pmatrix} \mapsto (p_1, \ldots, p_k)
\end{array}
$$

RI and $w_d > 0$ implies that $w_d = 1$

$$
\begin{aligned}
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\end{aligned}
$$

Concluding Remarks - Logics for Social Choice

Compare group decision postulates in terms of the language used to express them

M. Pauly. *On the Role of Language in Social Choice Theory*. Synthese, 163, 2, pgs. 227 - 243, 2008.

T. Daniëls. *Social choice and logic of simple games*. Journal of Logic and Computation, 21, 6, pgs. 883 - 906, 2011.

Concluding Remarks - Logics for Social Choice

Deriving Social Choice Theorems in a modal logic of profiles.

G. Ciná and U. Endriss. *Proving classical theorems of social choice theory in modal logic*. Journal of Autonomous Agents and Multiagent Systems, 30(5), 963-989, 2016.

N. Troquard, W. van der Hoek, and M. Wooldridge. *Reasoning about social choice functions*. Journal of Philosophical Logic 40(4), 473 - 498, 2011.

T. Agotnes, W. van der Hoek, and M. Wooldridge. *On the logic of preference and judgment aggregation*. Journal of Autonomous Agents and Multiagent Systems 22(1), 4 - 30, 2011.

Concluding Remarks - Logics for Social Choice

Verify existing proofs of Arrow's Theorem in higher-order logic proof assistants.

T. Nipkow. *Social choice theory in HOL: Arrow and Gibbard-Satterthwaite*. Journal of Automated Reasoning 43(3), 289 - 304, 2009.

F. Wiedijk. *Arrow's Impossibility Theorem*. Formalized Mathematics 15(4), 171 - 174, 2007.

Concluding Remarks - Logics for Social Choice

Classical first-order logic is sufficiently expressive to express all aspects of Arrow's Theorem (except that the set of agents in finite).

U. Grandi and U. Endriss. *First-order logic formalisation of impossibility theorems in preference aggregation*. Journal of Philosophical Logic 42(4), 595 - 618, 2013.

Arrow's Theorem for a fixed set of alternatives (e.g., $|N| = 2, |X| = 3$) can be embedded into classical propositional logic and automatically checked as a SAT problem. (The full theorem is proved by mathematical induction).

P. Tang and F. Lin. *Computer-aided proofs of Arrow's and other impossibility theorems*. Artificial Intelligence 173(11), 1041 - 1053, 2009.

Concluding Remarks - Logics for Social Choice

IIA + UD (independently of the Pareto principle or weakenings thereof) correspond to strong axioms about decisive coalitions. Prove representation theorems and develop a complete logic giving formal proofs of Arrow's and Wilson's Theorems (and related results).

W. Holliday and EP. *Arrow's Decisive Coalitions*. Social Choice and Welfare, 2018.

Concluding Remarks

- A number of different logics have been used to formalize various aspects of Arrow’s Theorem and related impossibility results. Which is best?

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Concluding Remarks

- A number of different logics have been used to formalize various aspects of Arrow's Theorem and related impossibility results. Which is best?
- The logical languages all have V, X as parameters.
- Dependence logic is a good match for Social Choice: it nicely captures reasoning about dependence (i.e., IIA) and independence (i.e., domain conditions).
- Separate reasoning about dependence/independence from reasoning about preferences or the structure of judgements.

D. Makinson. *Combinatorial versus decision-theoretic components of impossibility theorems*. Theory and Decision 40, 181 - 190, 1996.

Thank you!

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

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Binary Independence: For all profiles $\mathbf{R}, \mathbf{R}'$ and candidates $A, B \in X$:

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m**-Ary Independence**: For all profiles $\mathbf{R}, \mathbf{R}'$ and for all $S \subseteq X$ with $|S| = m$:

If $\mathbf{R}|_S = \mathbf{R}'|_S$, then $F(\mathbf{R})|_S = F(\mathbf{R}')|_S$

Weakening IIA

Theorem. (Blau) Suppose that $m = 2, \dots, |X| - 1$. If a social welfare function F satisfies m-ary independence, then it also satisfies binary independence.

J. Blau. *Arrow's theorem with weak independence*. Economica, 38, pgs. 413 - 420, 1971.

S. Cato. *Independence of Irrelevant Alternatives Revisited*. Theory and Decision, 2013.

Let $\mathcal{S} \subseteq \wp(X)$. F is $\mathcal{S}$**-independent** if for all profiles $\mathbf{R}, \mathbf{R}'$, and all $S \in \mathcal{S}$,

$$\text{if } \mathbf{R}|S = \mathbf{R}'|S, \text{ then } F(\mathbf{R})|S = F(\mathbf{R}')|S$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1,\ldots,k\}} S^j$$

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Theorem (Cato). (i) Suppose that $\mathcal{S} \subseteq \wp(X)$ is connected. If a collective choice rule F satisfies $\mathcal{S}$-independence, then it also satisfies binary independence.

(ii) Suppose that $\mathcal{S} \subseteq \wp(X)$ is not connected. Then, there exists a social welfare function F that satisfies $\mathcal{S}$-independence and weak Pareto but does not satisfy binary independence.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$:
if $A \; P_d \; B$, then $A \; P_{F(\mathbf{R})} \; B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$:
if $A \; P_d \; B$, then $B \; P_{F(\mathbf{R})} \; A$

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Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A\ I_{F(\mathbf{R})}\ B$

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if $A \; P_d \; B$, then $B \; P_{F(\mathbf{R})} \; A$

Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A \; I_{F(\mathbf{R})} \; B$

Non-Imposition: For all $A, B \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $A \; F(\mathbf{R}) \; B$

Weakening Unanimity

Theorem (Wilson) Suppose that N is a finite set. If a social welfare function satisfies universal domain, independence of irrelevant alternatives and non-imposition, then it is either null, dictatorial or inversely dictatorial.

R. Wilson. *Social Choice Theory without the Pareto principle*. Journal of Economic Theory, 5, pgs. 478 - 486, 1972.

Y. Murakami. *Logic and Social Choice*. Routledge, 1968.

S. Cato. *Social choice without the Pareto principle: A comprehensive analysis*. Social Choice and Welfare, 39, pgs. 869 - 889, 2012.

Thank you!